

On the equivalence of various definitions of mixed Poisson processes

D.P. Lyberopoulos *, N.D. Macheras and S.M. Tzaninis

July 20, 2016

Abstract

Under mild assumptions the equivalence of the mixed Poisson process with mixing parameter a real-valued random variable to the one with mixing distribution as well as to the mixed Poisson process in the sense of Huang is obtained, and a characterization of each one of the above mixed Poisson processes in terms of disintegrations is provided. Moreover, some examples of “canonical” probability spaces admitting counting processes satisfying the equivalence of all above statements are given.

Finally, it is shown that our assumptions are essential for the characterization of mixed Poisson processes in terms of disintegrations.

MSC 2010: Primary 60G55 ; secondary 60A10, 28A50, 60G05, 60K05, 60J27, 91B30.

Key Words: *mixed Poisson process, mixed renewal process, disintegration, Markov property.*

Introduction

To the best of our knowledge, given a probability space (Ω, Σ, P) , there are five definitions for mixed Poisson processes (MPPs for short). The first one, involving birth processes, traces back to Lundberg (cf. e.g. [6], page 61), while the second one is that of the *standard MPP with parameter a positive real-valued random variable* (cf. e.g. [6], Definition 4.3). Another definition of MPPs associated with a family $\{P_{\tilde{y}}\}_{\tilde{y} \in \tilde{\mathcal{Y}}}$ of probability measures on Σ and with a probability measure ν on the σ -algebra $\sigma(\{P_{\bullet}(E) : E \in \Sigma\})$ (written $\text{MPP}(\{P_{\tilde{y}}\}_{\tilde{y} \in \tilde{\mathcal{Y}}}, \nu)$ for short) is due to Huang [7], and is given in terms of inter-arrival processes, see Definition 2.2 (b). The other two definitions refer to the cases of a MPP with mixing parameter a real-valued random variable Θ ($\text{MPP}(\Theta)$ for short)

*The author is indebted to the Public Benefit Foundation ALEXANDER S. ONASSIS, which supported this research, under the Programme of Scholarships for Hellenes.

and a MPP with mixing distribution U (MPP(U) for short), see Definitions 2.2 (a) and (c), respectively.

The equivalence of Lundberg's definition to that of the MPP(U) is due to P. Albrecht [1], while the equivalence of the definition of the standard mixed Poisson process with parameter a positive real-valued random variable to that of the MPP(Θ) is due to R. F. Serfozo (see [13], page 290 together with [14], Theorem 3.1).

In this paper we first investigate the equivalence of the definitions of a MPP(Θ), a MPP(U) and a MPP($\{P_{\tilde{y}}\}_{\tilde{y} \in \tilde{\mathcal{Y}}}, \nu$). It is easy to see that a MPP(Θ) is always a MPP(U). But the inverse implication does not seem to be in general true, as it is not always possible (given a MPP(U)) to construct a real-valued random variable Θ such that $P_{\Theta} = U$, and a disintegration of P over U consistent with Θ .

In Section 2 we prove that given a counting process N , under Assumption 2.5 and under the assumption that P is perfect and Σ is countably generated, the existence of a real-valued random variable Θ such that N is a MPP(Θ) is equivalent with the existence of a distribution U on \mathfrak{B} such that N is a MPP(U), and the latter is equivalent with the existence of a family $\{P_{\tilde{y}}\}_{\tilde{y} \in \tilde{\mathcal{Y}}}$ of probability measures on Σ and of a probability measure ν on $\sigma(\{P_{\bullet}(E) : E \in \Sigma\})$ such that N is a MPP($\{P_{\tilde{y}}\}_{\tilde{y} \in \tilde{\mathcal{Y}}}, \nu$), see Theorem 2.6. In Theorem 2.7 we prove that under Assumption 2.4 and the existence of a disintegration of P over P_{Θ} consistent with a given real-valued random variable Θ , it follows that N is a MPP($\hat{\Theta}$) if and only if it is a MPP($P_{\hat{\Theta}}$) if and only if it is a MPP($\{Q_{\hat{\theta}}\}_{\hat{\theta} \in \mathbb{R}}, P_{\hat{\Theta}}$), where $\hat{\Theta}$ is a proper measurable function of Θ and $\{Q_{\hat{\theta}}\}_{\hat{\theta} \in \mathbb{R}}$ is a proper family of probability measures on Σ .

The proofs of Theorems 2.6 and 2.7 rely on two earlier results. The first one is due to Lyberopoulos and Macheras where it is proven that under the existence of an appropriate disintegration of P over P_{Θ} a MPP(Θ) can be reduced to an ordinary Poisson process under the disintegrating measures (see [8], Proposition 4.4). The second one is due to Macheras and Tzaninis where it is proven that under Assumption 2.4 within the class of mixed renewal processes, a counting process is a MPP(Θ) if and only if it has the P -Markov property (see [10], Theorem 2.11). For the definition of the P -Markov property we refer to e.g. [11], page 44.

In Section 3 we provide two examples of “canonical” probability spaces where all assumptions of Theorems 2.6 and 2.7 are valid. In particular, in both examples each of the assertions of Theorem 2.7 is valid.

Finally, in Section 4 we construct two counter-examples of non-trivial probability spaces where the characterization of MPPs in terms of disintegrations fails.

1 Preliminaries

By \mathbb{N} is denoted the set of all natural numbers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The symbol \mathbb{R} stands for the set of all real numbers, while $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ and \mathbb{R}^d denotes the Euclidean space of dimension $d \in \mathbb{N}$. Given a subset A of a set Ω we denote by A^c the complement $\Omega \setminus A$ of A and by χ_A the indicator function of A . For a map $f : D \rightarrow E$

we denote by R_f or by $f(D)$ the set $\{f(x) : x \in D\}$, and for a set $A \subseteq D$ we denote by $f \upharpoonright A$ the restriction of f to A , and by $f(A)$ the set $\{f(x) : x \in A\}$.

Given a probability space (Ω, Σ, P) a set $N \in \Sigma$ with $P(N) = 0$ is called a **P -null set**. For any two sets $A, B \in \Sigma$ we write $A =_P B$ if $P(A \Delta B) = 0$. Given a measurable space (\mathcal{Y}, H) , for any two Σ - H -measurable maps $X, Y : \Omega \mapsto \mathcal{Y}$ we write $X = Y$ P -a.s. if $\{X \neq Y\}$ is a P -null set.

Given a topology \mathfrak{T} on Ω write $\mathfrak{B}(\Omega)$ for its **Borel σ -algebra** on Ω , i.e. the σ -algebra generated by \mathfrak{T} and $\mathfrak{B} := \mathfrak{B}(\mathbb{R})$, $\overline{\mathfrak{B}} := \mathfrak{B}(\overline{\mathbb{R}})$, $\mathfrak{B}_d := \mathfrak{B}(\mathbb{R}^d)$ and $\mathfrak{B}_{\mathbb{N}} := \mathfrak{B}(\mathbb{R}^{\mathbb{N}})$ for the Borel σ -algebra of subsets of \mathbb{R} , $\overline{\mathbb{R}}$, \mathbb{R}^d and $\mathbb{R}^{\mathbb{N}}$ under the corresponding Euclidean topologies, respectively, while $\mathcal{L}^1(P)$ stands for the family of all real-valued P -integrable functions on Ω . Functions that are P -a.s. equal are not identified.

For the definitions of **real-valued random variables**, **random variables** and **random vectors** we refer to Cohn [2], pages 308 and 318.

Given two probability spaces (Ω, Σ, P) and (\mathcal{Y}, H, Q) as well as a Σ - H -measurable map $X : \Omega \mapsto \mathcal{Y}$ we denote by $\sigma(X) := \{X^{-1}(B) : B \in H\}$ the σ -algebra generated by X , while $\sigma(\{X_i\}_{i \in I}) := \sigma(\bigcup_{i \in I} \sigma(X_i))$ stands for the σ -algebra generated by a family $\{X_i\}_{i \in I}$ of Σ - H -measurable maps from Ω into \mathcal{Y} .

For any d -dimensional random vector X on Ω we apply the notation $P_X = \mathbf{K}(\theta)$ in the meaning that X is distributed according to the law $\mathbf{K}(\theta)$, where $\theta \in \mathbb{R}^d$. In particular, $\mathbf{P}(\theta)$ and $\mathbf{Exp}(\theta)$, where θ is positive parameter, stand for the law of Poisson and exponential distribution, respectively (cf. e.g. [11]).

We write $\mathbb{E}[X|\mathcal{F}]$ for a conditional expectation of X given \mathcal{F} (see [2], page 342 for the definition). For $X := \chi_E \in \mathcal{L}^1(P)$ with $E \in \Sigma$ we set $P(E | \mathcal{F}) := \mathbb{E}_P[\chi_E | \mathcal{F}]$.

Given a real-valued random variable X on Ω and a random vector $\Theta : \Omega \mapsto \mathbb{R}^d$, a conditional distribution of X over Θ is a map $P_{X|\Theta}$ from $\mathfrak{B} \times \Omega$ into $[0, 1]$ such that

(cd1) for each $\omega \in \Omega$ the set-function $P_{X|\Theta}(\bullet, \omega)$ is a probability measure on \mathfrak{B} ;

(cd2) for each $B \in \mathfrak{B}$ we have

$$P_{X|\Theta}(B, \bullet) = P(\Theta^{-1}(B) | \sigma(\Theta)) \quad P \upharpoonright \sigma(\Theta)\text{-a.s.},$$

where $P_{X|\Theta}(B, \bullet)$ is $\sigma(\Theta)$ -measurable for any fixed $B \in \mathfrak{B}$.

For simplicity we write $k := P_{X|\Theta}$ and define the map $K(\Theta)$ from $\mathfrak{B} \times \Omega$ into $[0, 1]$ by means of

$$K(\Theta)(B, \omega) := (k(B, \bullet) \circ \Theta)(\omega) \quad \forall B \in \mathfrak{B} \quad \forall \omega \in \Omega.$$

Then for $\theta = \Theta(\omega)$ with $\omega \in \Omega$ the probability measures $k(\bullet, \theta)$ are distributions on \mathfrak{B} and so we may write $\mathbf{K}(\theta)(\bullet)$ instead of $k(\bullet, \theta)$. Consequently, in this case $K(\Theta)$ will be written by $\mathbf{K}(\Theta)$.

For any real-valued random variables X, Y on Ω we say that $P_{X|\Theta}$ and $P_{Y|\Theta}$ are $P \upharpoonright \sigma(\Theta)$ -equivalent and we write $P_{X|\Theta} = P_{Y|\Theta}$ $P \upharpoonright \sigma(\Theta)$ -a.s., if there exists a P -null set $N \in \sigma(\Theta)$ such that for any $\omega \notin N$ and $B \in \mathfrak{B}$ the equality $P_{X|\Theta}(B, \omega) = P_{Y|\Theta}(B, \omega)$ holds true.

From now on (Ω, Σ, P) is a fixed probability space, while $(\mathcal{Y}, H) := (\mathbb{R}, \mathfrak{B})$, $(\Xi, Z) := (\mathbb{R}^d, \mathfrak{B}_d)$.

2 The results

We first recall some additional background material, needed in this section.

A family $N := \{N_t\}_{t \in \mathbb{R}_+}$ of random variables from (Ω, Σ) into $(\overline{\mathbb{R}}, \overline{\mathfrak{B}})$ is called a **counting process** if there exists a P -null set $\Omega_N \in \Sigma$ such that the process N restricted on $\Omega \setminus \Omega_N$ takes values in $\mathbb{N}_0 \cup \{\infty\}$, has right-continuous paths, presents jumps of size (at most) one, vanishes at $t = 0$ and increases to infinity. Denote by $T := \{T_n\}_{n \in \mathbb{N}_0}$ and $W := \{W_n\}_{n \in \mathbb{N}}$ the **arrival process** and **interarrival process** respectively (cf. e.g. [11], Section 1.1, page 6 for the definition) associated with N .

Recall that for a random vector $\Theta : \Omega \mapsto \mathbb{R}^d$ a family $\{X_i\}_{i \in I}$ of real-valued random variables X_i on Ω

- is **P -conditionally (stochastically) independent given Θ** , if for each $n \in \mathbb{N}$ with $n \geq 2$ we have

$$P\left(\bigcap_{j=1}^n \{X_{i_j} \leq x_{i_j}\} \mid \sigma(\Theta)\right) = \prod_{k=1}^n P(\{X_{i_k} \leq x_{i_k}\} \mid \sigma(\Theta)) \quad P \upharpoonright \sigma(\Theta) - \text{a.s.}$$

whenever i_1, \dots, i_n are distinct members of I and $(x_{i_1}, \dots, x_{i_n}) \in \mathbb{R}^n$;

- is **P -conditionally identically distributed given Θ** , if

$$P(F \cap X_i^{-1}(B)) = P(F \cap X_j^{-1}(B))$$

whenever $i, j \in I$, $F \in \sigma(\Theta)$ and $B \in \mathfrak{B}$.

For the rest of the paper we simply write “conditionally” in the place of “conditionally given Θ ” whenever Θ is clear from the context.

Definition 2.1 Let Q be a probability measure on \mathfrak{B}_d . A family $\{P_\theta\}_{\theta \in \mathbb{R}^d}$ of probability measures on Σ is called a **disintegration** of P over Q if

(d1) for each $D \in \Sigma$ the map $\theta \mapsto P_\theta(D)$ is \mathfrak{B}_d -measurable;

(d2) $\int P_\theta(D) Q(d\theta) = P(D)$ for each $D \in \Sigma$.

If $\Theta : \Omega \mapsto \mathbb{R}^d$ is an inverse-measure-preserving random vector (i.e. $P_\Theta(B) = Q(B)$ for each $B \in \mathfrak{B}_d$), a disintegration $\{P_\theta\}_{\theta \in \mathbb{R}^d}$ of P over Q is called **consistent** with Θ if, for each $B \in \mathfrak{B}_d$, the equality $P_\theta(\Theta^{-1}(B)) = 1$ holds for Q -almost every $\theta \in B$.

Remark. If Σ is countably generated (cf. e.g. [2], Section 3.4, page 102 for the definition) and P is perfect (see [3], p. 291 for the definition), then there always exists a disintegration $\{P_\theta\}_{\theta \in \mathbb{R}^d}$ of P over Q consistent with any inverse-measure-preserving random vector $\Theta : \Omega \mapsto \mathbb{R}^d$ (see [3], Theorems 6 and 3) and this means, that in most cases appearing in applications (e.g. Polish spaces) disintegrations consistent with a random vector exist.

Throughout what follows, unless stated otherwise, $N := \{N_t\}_{t \in \mathbb{R}_+}$ is a counting process, $T := \{T_n\}_{n \in \mathbb{N}_0}$ is an arrival process, $W := \{W_n\}_{n \in \mathbb{N}}$ is its induced interarrival process and without loss of generality we may and do assume that $\Omega_N = \emptyset$.

A Poisson process N with respect to P with parameter $\theta > 0$ is denoted by $P\text{-PP}(\theta)$.

Definitions 2.2 A counting process N is:

(a) a **mixed Poisson process with mixing parameter a real-valued random variable** Θ such that $P_\Theta((0, \infty)) = 1$ (written $P\text{-MPP}(\Theta)$ for short), if it has P -conditionally independent and P -conditionally stationary increments (cf. e.g. [11], Section 4.1, page 86 for the definition) and condition

$$\forall t \in (0, \infty) \quad [P_{N_t|\Theta} = \mathbf{P}(\Theta t) \quad P \upharpoonright \sigma(\Theta) - \text{a.s.}]$$

holds true (cf. e.g. [11], page 87);

(b) a **mixed Poisson process associated with** $\{P_{\tilde{y}}\}_{\tilde{y} \in \tilde{\mathcal{Y}}}$ **and** ν ($P\text{-MPP}(\{P_{\tilde{y}}\}_{\tilde{y} \in \tilde{\mathcal{Y}}}, \nu)$ for short), if for every $r \in \mathbb{N}$ and for all $w_1, \dots, w_r > 0$ condition

$$P\left(\bigcap_{k=1}^r \{W_k \leq w_k\}\right) = \int \prod_{k=1}^r P_{\tilde{y}}(\{W_k \leq w_k\}) \nu(d\tilde{y}),$$

holds true, where $\{P_{\tilde{y}}\}_{\tilde{y} \in \tilde{\mathcal{Y}}}$ is a family of probability measures on Σ and ν is a probability measure on $B(\Sigma) := B(\tilde{\mathcal{Y}}, \Sigma) := \sigma(\{P_\bullet(E) : E \in \Sigma\})$ such that W is $P_{\tilde{y}}$ -independent and $(P_{\tilde{y}})_{W_n} = \mathbf{Exp}(\alpha(\tilde{y}))$ for every $n \in \mathbb{N}$ and for ν -a.a. $\tilde{y} \in \tilde{\mathcal{Y}}$, where α is a positive measurable function on \mathbb{R} (see [7], page 2);

(c) a **mixed Poisson process with mixing distribution** $U : \mathfrak{B}((0, \infty)) \mapsto [0, 1]$ (written $P\text{-MPP}(U)$ for short) if

$$P\left(\bigcap_{j=1}^m \{N_{t_j} - N_{t_{j-1}} = \kappa_j\}\right) = \int_{(0, \infty)} \prod_{j=1}^m e^{-\theta(t_j - t_{j-1})} \frac{(\theta(t_j - t_{j-1}))^{\kappa_j}}{\kappa_j!} U(d\theta)$$

holds for all $m \in \mathbb{N}$ and $t_0, t_1, \dots, t_m \in \mathbb{R}_+$ with $0 = t_0 < t_1 < \dots < t_m$ and for all $\kappa_j \in \mathbb{N}_0$, $j \in \{1, \dots, m\}$ (cf. e.g. [12], page 9).

The following definition has been introduced in [10], Definitions 2.3.

Definition 2.3 A counting process N is called an **extended MRP with mixing parameters** Θ **and** h , **and interarrival time conditional distribution** $\mathbf{K}(h(\Theta))$ (written $P\text{-eMRP}(\mathbf{K}(h(\Theta)))$ for short), if h is a \mathbb{R}^k -valued $\mathfrak{B}(D)$ - \mathfrak{B}_k -measurable function on $D \in \mathfrak{B}_d$ with $R_\Theta \subseteq D$ for $k \in \mathbb{N}$, if the induced interarrival process W is P -conditionally independent and

$$\forall n \in \mathbb{N} \quad [P_{W_n|\Theta} = \mathbf{K}(h(\Theta)) \quad P \upharpoonright \sigma(\Theta) - \text{a.s.}].$$

In particular, if $k = d$ and $h = id_{\mathbb{R}^d}$ then N is a **P -MRP with interarrival time distribution** $\mathbf{K}(\Theta)$ (written $P\text{-MRP}(\mathbf{K}(\Theta))$ for short). Moreover if $h = id_{\mathbb{R}^d}$

and if there exists a $\theta_0 \in \mathbb{R}^d$ with $P(\{\Theta = \theta_0\}) = 1$ then N is a **P -renewal process with interarrival time distribution $\mathbf{K}(\theta_0)$** (written $P\text{-RP}(\mathbf{K}(\theta_0))$ for short).

Without loss of generality we may and do assume that

$$(1) \quad \forall n \in \mathbb{N} \quad [P_{W_n|\Theta} = \mathbf{K}(h(\Theta))].$$

From now on, unless stated otherwise, Θ is a positive real-valued random variable on Ω .

The following assumption is a special case of Assumption 2.6 from [10].

Assumption 2.4 Let $D \in \mathfrak{B}$ with $R_\Theta \subseteq D$, $h : D \mapsto \mathbb{R}$ be a $\mathfrak{B}(D)$ -measurable function, let N be a P -eMRP($\mathbf{K}(h(\Theta))$) and let $\{P_\theta\}_{\theta \in D}$ be a disintegration of P over P_Θ consistent with Θ . It follows by [9], Lemma 3.5 together with condition (1) that

$$(2) \quad \forall n \in \mathbb{N} \quad \forall \theta \in D \quad [(P_\theta)_{W_n} = \mathbf{K}(h(\theta))].$$

For any $\theta \in D$ and $t \in \mathbb{R}_+$ put

$$F_{h(\theta)}(t) := P_\theta(\{W_n \leq t\}) \quad \text{for all } n \in \mathbb{N}.$$

Clearly the function $F_{h(\theta)}$ depends on the distribution of W_n and, because of condition (2), on h . We say that N , h and $\{P_\theta\}_{\theta \in D}$ satisfy Assumption 2.4, if there exists a P_Θ -null set $L_h := L_{h,N,\{P_\theta\}_{\theta \in D}}$ in $\mathfrak{B}(D)$ such that for any $\theta \notin L_h$ the function $F_{h(\theta)}$ is continuously differentiable on $(0, \infty)$, there exists a function $C \in \mathcal{L}^1(P_{h(\Theta)})$ with $0 < F'_{h(\theta)}(t) < C(h(\theta))$ for each $t > 0$, and the function $p_h : D \setminus L_h \mapsto \mathbb{R}$ defined by means of $p_h(\theta) := p_{h,1}(\theta) := \lim_{t \rightarrow 0} F'_{h(\theta)}(t)$ is positive and injective.

For the special case $D = \mathbb{R}$ and $h := id_{\mathbb{R}}$ we write for simplicity L , F_θ and p_1 in the place of L_h , $F_{h(\theta)}$ and p_h respectively, and we say that N and $\{P_\theta\}_{\theta \in \mathbb{R}}$ satisfy Assumption 2.4.

Assumption 2.5 Given a counting process N there exists a real-valued random variable Θ on Ω and disintegration $\{P_\theta\}_{\theta \in D}$ of P over P_Θ consistent with Θ satisfying together with N Assumption 2.4.

Theorem 2.6 For a counting process N consider the following assertions:

- (i) there exists a real-valued random variable $\check{\Theta}$ on Ω such that N is a P -MPP($\check{\Theta}$) with respect to P ;
- (ii) there exists a family $\{P_{\check{y}}\}_{\check{y} \in \check{\mathcal{Y}}}$ of probability measures on Σ and a probability measure ν on $B(\Sigma)$ such that N is a P -MPP($\{P_{\check{y}}\}_{\check{y} \in \check{\mathcal{Y}}}, \nu$);
- (iii) there exists a real-valued random variable $\check{\Theta}$ on Ω and a disintegration $\{Q_{\check{\theta}}\}_{\check{\theta} \in \check{\mathbb{R}}}$ of P over $P_{\check{\Theta}}$ consistent with $\check{\Theta}$ such that N is a $Q_{\check{\theta}}$ -PP($\check{\theta}$) for $P_{\check{\Theta}}$ -a.a. $\check{\theta} \in \check{\mathbb{R}}$;

(iv) there exists a distribution $U : \mathfrak{B} \mapsto [0, 1]$ with $U((0, \infty)) = 1$ such that N is a P -MPP(U).

Then (iii) \implies (i) \implies (iv).

Moreover, if P is perfect and Σ is countably generated then statements (i) and (iii) are equivalent and (i) \implies (ii).

If in addition, Assumption 2.5 holds true, then all statements (i) to (iv) are equivalent.

Proof. First note that implication (iii) \implies (i) is immediate by Proposition 4.4 of [8], while the implication (i) \implies (iv) follows by an easy computation.

Assume now that P is perfect and Σ is countably generated. Then the equivalence (i) \iff (iii) follows by [8], Proposition 4.4, since under this assumption for any real-valued random variable $\tilde{\Theta}$ on Ω there always exists a disintegration $\{Q_{\tilde{\theta}}\}_{\tilde{\theta} \in \mathbb{R}}$ of P over $P_{\tilde{\Theta}}$ consistent with $\tilde{\Theta}$ (see [3], Theorems 6 and 3).

Ad (i) \implies (ii): If (i) is true, since (i) is equivalent with (iii), it follows that there exists a disintegration $\{Q_{\tilde{\theta}}\}_{\tilde{\theta} \in \mathbb{R}}$ of P over $P_{\tilde{\Theta}}$ consistent with $\tilde{\Theta}$ such that N is a $Q_{\tilde{\theta}}$ -PP($\tilde{\theta}$) for $P_{\tilde{\Theta}}$ -a.a. $\tilde{\theta} \in \mathbb{R}$. Then applying [10], Proposition 2.2, we obtain (ii) for $\{P_{\tilde{y}}\}_{\tilde{y} \in \tilde{Y}} := \{Q_{\tilde{\theta}}\}_{\tilde{\theta} \in \mathbb{R}}$ and $\nu := P_{\tilde{\Theta}}$.

Assume now in addition that Assumption 2.5 holds true.

Ad (ii) \implies (i): If assertion (ii) holds true, then by Definition 2.2 (b) together with e.g. [11], Theorem 2.3.4, we obtain that N is a $P_{\tilde{y}}$ -PP($\alpha(\tilde{y})$) for ν -a.a. $\tilde{y} \in \tilde{Y}$; hence applying [11], Lemma 2.3.1 we deduce that for ν -a.a. $\tilde{y} \in \tilde{Y}$ the equality

$$(3) \quad P_{\tilde{y}}\left(\bigcap_{j=1}^m \{N_{t_j} - N_{t_{j-1}} = n_j\}\right) = \frac{n!}{\prod_{j=1}^m n_j!} \cdot \prod_{j=1}^m \left(\frac{t_j - t_{j-1}}{t_m}\right)^{n_j} \cdot P_{\tilde{y}}(\{N_{t_m} = n\})$$

holds true for any $m \in \mathbb{N}$, any $t_0, t_1, \dots, t_m \in \mathbb{R}_+$ such that $0 = t_0 < t_1 < \dots < t_m$ and any $n_1, \dots, n_m \in \mathbb{N}_0$ such that $\sum_{j=1}^m n_j = n$. The latter implies again for ν -a.a. $\tilde{y} \in \tilde{Y}$ the equality

$$(4) \quad P_{\tilde{y}}(\{N_s = k\} \cap \{N_t - N_s = n - k\}) = \binom{n}{k} \cdot \left(\frac{s}{t}\right)^k \cdot \left(1 - \frac{s}{t}\right)^{n-k} \cdot P_{\tilde{y}}(\{N_t = n\})$$

for all $s, t \in (0, \infty)$ such that $s < t$ and all $k, n \in \mathbb{N}_0$ such that $k \leq n$. Putting $\mathcal{F}_N := \sigma(\{N_t\}_{t \in \mathbb{R}_+})$, $\mathcal{F}_W := \sigma(\{W_n\}_{n \in \mathbb{N}})$ and $\mathcal{F}_T := \sigma(\{T_n\}_{n \in \mathbb{N}_0})$ we then get $\mathcal{F}_T = \mathcal{F}_W$ and $\mathcal{F}_T = \mathcal{F}_N$ (cf. e.g. [11], Lemmas 1.1.1 and 2.1.3 respectively).

Claim. The family $\{P_{\tilde{y}}\}_{\tilde{y} \in \tilde{Y}}$ of probability measures is a disintegration of $P \upharpoonright \mathcal{F}_N$ over ν .

Proof. Since $\mathcal{F}_N = \mathcal{F}_W$, it is sufficient to show that $\{P_{\tilde{y}}\}_{\tilde{y} \in \tilde{Y}}$ is a disintegration of $P \upharpoonright \mathcal{F}_W$ over ν .

It follows by Definition 2.2 (b) that the property (d1) holds true; hence it is enough to show (d2) for any $E \in \mathcal{F}_W$. Put $\mathcal{G} := \bigcup_{n \in \mathbb{N}} \sigma(W_n)$. Due to Definition 2.2 (b) we have that (d2) is satisfied for each $\{W_n \leq w_n\}$ where $w_n > 0$ for any $n \in \mathbb{N}$.

Denote by \mathcal{G}_\cap be the generator of \mathcal{F}_W consisting of \mathcal{G} and all finite intersections of elements of \mathcal{G} and put

$$\mathcal{D} := \{E \in \mathcal{F}_W : P(E) = \int P_{\tilde{y}}(E) \nu(d\tilde{y})\}.$$

Then it easy to prove that the family \mathcal{D} is a Dynkin class containing \mathcal{G}_\cap ; hence by the Monotone Class Theorem we get that $\mathcal{D} = \mathcal{F}_W$ and the above claim follows. \square

Fix on arbitrary $m \in \mathbb{N}$, $t_0, t_1, \dots, t_{m+1} \in \mathbb{R}_+$ such that $0 = t_0 < t_1 < \dots < t_{m+1}$ and $n_0, n_1, \dots, n_{m+1} \in \mathbb{N}_0$ such that $0 = n_0 \leq n_1 \leq \dots \leq n_{m+1}$. Using the above claim and the equalities (3) and (4) we get by standard computations that

$$\begin{aligned} P\left(\bigcap_{j=1}^m \{N_{t_j} = n_j\}\right) \cdot P(\{N_{t_m} = n_m\} \cap \{N_{t_{m+1}} = n_{m+1}\}) \\ = P\left(\bigcap_{j=1}^{m+1} \{N_{t_j} = n_j\}\right) \cdot P(\{N_{t_m} = n_m\}), \end{aligned}$$

or equivalently that N has the P -Markov property. Since N has the P -Markov property and Assumption 2.5 is valid, we may apply [10], Proposition 2.7, to obtain assertion (i).

Ad (iv) \implies (i): If (iv) is valid then by Theorem 4.2 from [12], N has the P -Markov property. By Assumption 2.5 there exists a real-valued random variable Θ on Ω and a disintegration $\{P_\theta\}_{\theta \in D}$ of P over P_Θ consistent with Θ satisfying together with N Assumption 2.4. Thus, we may apply [10] Proposition 2.7 to get (i). \square

Theorem 2.7 *Let N be a P -eMRP($\mathbf{K}(h(\Theta))$) and let $\{P_\theta\}_{\theta \in D}$ be a disintegration of P over P_Θ consistent with Θ satisfying together with N and h Assumption 2.4. Suppose that there exists a P_Θ -null set $L_0 \in \mathfrak{B}(D)$ such that $h \upharpoonright D \setminus L_0$ is injective. Put $O_h := L_0 \cup L_h$ and $\hat{\Theta}(\omega) := (p_h \circ \Theta)(\omega)$ if $\omega \in \Theta^{-1}(D \setminus O_h)$, where L_h and p_h are as in Assumption 2.4, and denote again by $\hat{\Theta}$ any measurable extension of $\hat{\Theta}$ from $\Theta^{-1}(D \setminus O_h)$ to Ω . For any fixed $A \in \Sigma$ put*

$$Q_{\hat{\Theta}}(A) := \begin{cases} (P_\bullet(A) \circ p_h^{-1})(\hat{\theta}) & \text{if } \hat{\theta} \in p_h(D \setminus O_h); \\ P(A) & \text{otherwise.} \end{cases}$$

Then $\{Q_{\hat{\theta}}\}_{\hat{\theta} \in \mathbb{R}}$ is a disintegration of P over $P_{\hat{\Theta}}$ consistent with $\hat{\Theta}$, and the following are equivalent:

- (i) N is a P -MPP($\hat{\Theta}$);
- (ii) N is a P -MPP($\{Q_{\hat{\theta}}\}_{\hat{\theta} \in \mathbb{R}}, P_{\hat{\Theta}}$);
- (iii) N is a $Q_{\hat{\theta}}$ -PP($\hat{\theta}$) for $P_{\hat{\Theta}}$ -a.a. $\hat{\theta} \in \mathbb{R}$;

(iv) N is a P -MPP($P_{\hat{\Theta}}$).

Proof. The fact that $\{Q_{\hat{\theta}}\}_{\hat{\theta} \in \mathbb{R}}$ is a disintegration of P over $P_{\hat{\Theta}}$ consistent with $\hat{\Theta}$ is a consequence of [10], Lemma 2.4.

The equivalence (i) \iff (iii) is due to [8], Proposition 4.4.

Ad (i) \implies (ii) : Since assertion (i) holds true and $\{Q_{\hat{\theta}}\}_{\hat{\theta} \in \mathbb{R}}$ is a disintegration of P over $P_{\hat{\Theta}}$ consistent with $\hat{\Theta}$, it follows by [10], Proposition 2.2 that (ii) is valid.

Ad (ii) \implies (i) : If (ii) holds true, we get as in the proof of Theorem 2.6, (ii) \implies (i), that N has the P -Markov property; hence by [10], Theorem 2.11, we obtain (i).

The implication (i) \implies (iv) follows by an easy computation.

Ad (iv) \implies (i) : If (iv) is valid then N has the P -Markov property (see [12], Theorem 4.2). So, by [10], Theorem 2.11, assertion (i) follows. \square

Remarks 2.8 (a) If the assumptions of Theorem 2.7 are satisfied and if one of its assertions (i)-(iv) is valid, then p_h and h coincide outside the P_{Θ} -null set $\tilde{L}_3 \cup O_h$, and $\hat{\Theta}$ and $\tilde{\Theta}$ coincide outside of the P -null set $\Theta^{-1}(O_h)$.

In fact, let $\{Q_{\hat{\theta}}\}_{\hat{\theta} \in \mathbb{R}}$ be as in Theorem 2.7 and assume that (iii) holds true. It then follows that $(Q_{\hat{\theta}})_{W_n} = \mathbf{Exp}(\hat{\theta})$ and that W is $Q_{\hat{\theta}}$ -independent for any $\hat{\theta} \in p_h(D \setminus O_h)$ (cf. e.g. [11], Theorem 2.3.4). Applying now [10], Lemma 2.4, for $p_h(O_h)$ and p_h in the place of L_0 and h respectively, we obtain that $(P_{\theta})_{W_n} = \mathbf{Exp}(p_h(\theta))$ and that W is P_{θ} -independent for any $\theta \in D \setminus O_h$. Since N is P -eMRP($\mathbf{K}(h(\Theta))$), due to [10], Lemma 2.9, we can find a P_{Θ} -null set $\tilde{L}_3 \in \mathfrak{B}(D)$ such that $(P_{\theta})_{W_n} = \mathbf{K}(h(\theta))$ for any $\theta \in D \setminus \tilde{L}_3$ and $n \in \mathbb{N}$.

As a consequence, we deduce that for any $\theta \in D \setminus (\tilde{L}_3 \cup O_h)$ and $\hat{\theta} = p_h(\theta)$ conditions

$$\mathbf{Exp}(\hat{\theta}) = (Q_{\hat{\theta}})_{W_n} = (P_{\theta})_{W_n} = \mathbf{K}(h(\theta))$$

hold true, implying that $p_h(\theta) = h(\theta)$ for any $\theta \in D \setminus (\tilde{L}_3 \cup O_h)$.

(b) It is worth noticing that if all statements (i) to (iv) of Theorem 2.7 are equivalent and if one of them is valid, then its assumptions are necessary. More precisely, let h and L_0 be as in Theorem 2.7 such that $\mathbb{E}[h(\Theta)] < \infty$, let N be a counting process and $\tilde{\Theta} := h \circ \Theta$. Because of (a) we may take h and $\tilde{\Theta}$ in the place of p_h and $\hat{\Theta}$ respectively. Assume that assertions (i)-(iv) of Theorem 2.7 are all equivalent and each of them is valid with $\tilde{\Theta}$, h and L_0 in the place of $\hat{\Theta}$, p_h and O_h . Then N is a P -eMRP($\mathbf{K}(h(\Theta))$), and there exists a disintegration $\{P_{\theta}\}_{\theta \in D}$ of P over P_{Θ} consistent with Θ satisfying together with N and h Assumption 2.4.

In fact, since (iii) is valid, there exists a disintegration $\{Q_{\tilde{\theta}}\}_{\tilde{\theta} \in \mathbb{R}}$ of P over $P_{\tilde{\Theta}}$ consistent with $\tilde{\Theta}$ such that the counting process N is a PP($\tilde{\theta}$) with respect to $Q_{\tilde{\theta}}$ for any $\tilde{\theta} \in h(D \setminus L_0)$. The latter is equivalent with the fact that $(Q_{\tilde{\theta}})_{W_n} = \mathbf{Exp}(\tilde{\theta})$ and that W is $Q_{\tilde{\theta}}$ -independent for any $\tilde{\theta} \in h(D \setminus L_0)$ (cf. e.g. [11], Theorem 2.3.4). For any $\theta \in D$ and $A \in \Sigma$ define

$$P_{\theta}(A) := \begin{cases} (Q_{\bullet}(A) \circ h)(\theta) & \text{if } \theta \in D \setminus L_0; \\ P(A) & \text{otherwise.} \end{cases}$$

Applying [10], Lemma 2.4, we obtain that $\{P_\theta\}_{\theta \in D}$ of P over P_Θ consistent with Θ and that $(P_\theta)_{W_n} = \mathbf{Exp}(h(\theta))$ and W is P_θ -independent for any $\theta \in D \setminus L_0$. Applying now [10], Lemma 2.9, together with [9], Lemma 3.6, we obtain that $P_{W_n|\Theta} = \mathbf{Exp}(h(\Theta))$ $P \upharpoonright \sigma(\Theta)$ -a.s. and that W is P -conditionally independent; hence N is a P -eMRP($\mathbf{K}(h(\Theta))$). It remains to show that $\{P_\theta\}_{\theta \in D}$, N and h satisfy Assumptions 2.4.

In fact, for any $\theta \in D \setminus L_0$, $t \in \mathbb{R}_+$ and $n \in \mathbb{N}$ put $F_{h(\theta)}(t) := P_\theta(\{W_n \leq t\}) := 1 - e^{-h(\theta)t}$. Clearly, $F_{h(\theta)}$ is continuously differentiable on $(0, \infty)$. Define the map $C \in \mathcal{L}_1(P_{h(\Theta)})$ by $C(h(\theta)) := h(\theta)$ for any $\theta \in D \setminus L_0$, and for any fixed $\theta \in D \setminus L_0$ define the density $f_{h(\theta)} := F'_{h(\theta)}$ by $f_{h(\theta)}(t) := h(\theta) \cdot e^{-h(\theta)t}$ for any $t > 0$. Clearly, for any fixed $\theta \in D \setminus L_0$, the density $f_{h(\theta)}$ is dominated by $C(h(\theta))$, and the function $\lim_{t \rightarrow 0} f_{h(\theta)}(t) = h(\theta)$ is positive and injective; hence $\{P_\theta\}_{\theta \in D}$, N and h satisfy Assumptions 2.4.

3 Examples

By $(\Omega \times \mathcal{Y}, \Sigma \otimes H, P \otimes Q)$ is denoted the product probability space of (Ω, Σ, P) and (\mathcal{Y}, H, Q) , and by π_Ω and $\pi_\mathcal{Y}$ the canonical projections from $\Omega \times \mathcal{Y}$ onto Ω and \mathcal{Y} , respectively.

In this section we first provide an example of “canonical” probability spaces admitting extended MRPs. Next we present, as special cases, two examples of probability spaces satisfying all assumptions of Theorems 2.6 and 2.7. In particular, in both examples each of the assertions of Theorems 2.6 and 2.7 is valid.

Throughout what follows, we put $\mathcal{Y} := (0, \infty)$, $H := \mathfrak{B}(\mathcal{Y})$, $\tilde{\Omega} := \mathcal{Y}^\mathbb{N}$, $\Omega := \tilde{\Omega} \times G$ for $G \in \mathfrak{B}$, $\tilde{\Sigma} := \mathfrak{B}(\tilde{\Omega})$ and $\Sigma := \mathfrak{B}(\Omega) = \mathfrak{B}(\tilde{\Omega}) \otimes \mathfrak{B}(G)$ for simplicity.

The next example is a special case of Example 3.1 from [10].

Example 3.1 Let μ be an arbitrary probability measure on $\mathfrak{B}(G)$ and let $Q_n(\theta)$ be probability measures on $\mathfrak{B}(\mathcal{Y})$ for all $n \in \mathbb{N}$ and for any fixed $\theta \in G$, which is absolutely continuous with respect to Lebesgue measure λ on \mathfrak{B} . Moreover, suppose that there exists a $\mathfrak{B}(G)$ -measurable function $h : G \mapsto \mathbb{R}$ such that $Q_n(\theta) = \mathbf{K}(h(\theta))$ for any $n \in \mathbb{N}$ and $\theta \in G$, where for any $B \in \mathfrak{B}(\mathcal{Y})$ the function $\mathbf{K}(h(\bullet))(B) : G \mapsto \mathbb{R}$ is $\mathfrak{B}(G)$ -measurable. Put $\tilde{P}_\theta := \otimes_{n \in \mathbb{N}} Q_n(\theta)$ for any $\theta \in G$. Define the set-function $P(E) := \int \tilde{P}_\theta(E^\theta) \mu(d\theta)$, for each $E \in \Sigma$, where E^θ is the θ -section of E , and put $P_\theta := \tilde{P}_\theta \otimes \delta_\theta$ for any $\theta \in G$, where δ_θ is the Dirac measure at θ . Then P is a probability measure on Σ and $\{P_\theta\}_{\theta \in G}$ is a disintegration of P over μ consistent with the canonical projection π_G from Ω onto G (compare [10], Example 3.1).

Clearly, putting $\Theta := \pi_G$ we get $P_\Theta = \mu$. Set $W_n := \pi_n$ for any $n \in \mathbb{N}$, where $\pi_n : \Omega \mapsto \mathcal{Y}$ is the canonical projection, and $W := \{W_n\}_{n \in \mathbb{N}}$. Put $T_n := \sum_{k=1}^n W_k$ for any $n \in \mathbb{N}_0$ and $T := \{T_n\}_{n \in \mathbb{N}_0}$, and let $N := \{N_t\}_{t \in \mathbb{R}_+}$ be the counting process induced by T by means of $N_t := \sum_{n=1}^\infty \chi_{\{T_n \leq t\}}$ for all $t \in \mathbb{R}_+$ (cf. e.g [11], Theorem 2.1.1). Applying the same arguments as in [10], Example 3.1, we get that N is a P -eMRP($\mathbf{K}(h(\Theta))$).

In the next example the real-valued random variable $\hat{\Theta}$ is distributed according to the Gamma law, a common choice in Risk Theory.

Example 3.2 Let $G := \mathcal{Y}$, let $\xi = \mathbf{IGa}(\alpha, \beta)$, with $\alpha, \beta > 0$ be a probability measure on $\mathfrak{B}(\mathcal{Y})$ i.e.

$$\xi(B) := \int_B \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot t^{-\alpha-1} \cdot e^{-\frac{\beta}{t}} \cdot \chi_{(0,\infty)}(t) \lambda(dt) \quad \text{for each } B \in \mathfrak{B}(\mathcal{Y})$$

and let $h : \mathcal{Y} \mapsto \mathbb{R}$ be defined by $h(\theta) := \frac{1}{\theta}$ for any $\theta \in \mathcal{Y}$. Fix now on arbitrary $\theta \in \mathcal{Y}$ and define the probability measures $Q_n(\theta)$ by means of $Q_n(\theta) := \mathbf{Exp}(h(\theta))$ for all $n \in \mathbb{N}$. Let (Ω, Σ, P) , Θ , N , W and $\{P_\theta\}_{\theta \in \mathcal{Y}}$ be as in Example 3.1 with $G = \mathcal{Y}$ and ξ in the place of μ .

Define the map $C \in \mathcal{L}^1(P_{h(\Theta)})$ by $C(h(\theta)) := h(\theta)$ for any $\theta \in \mathcal{Y}$, and for any fixed $\theta \in \mathcal{Y}$ define the density $f_{h(\theta)} := F'_{h(\theta)}$ by $f_{h(\theta)}(t) := h(\theta) \cdot e^{-h(\theta)t}$ for any $t > 0$. Clearly, for any fixed $\theta \in \mathcal{Y}$, the density $f_{h(\theta)}$ is dominated by $C(h(\theta))$, and the function $p_h : \mathcal{Y} \mapsto \mathcal{Y}$ defined by means of $p_h(\theta) := \lim_{t \rightarrow 0} f_{h(\theta)}(t) = h(\theta)$ for any $\theta \in \mathcal{Y}$, is positive and injective; hence $\{P_\theta\}_{\theta \in \mathcal{Y}}$, N and h satisfy Assumption 2.4.

Let $\hat{\Theta} := h \circ \Theta$ and put $Q_{\hat{\theta}}(E) := (P_\bullet(E) \circ h^{-1})(\hat{\theta})$ for any $\hat{\theta} > 0$ and $E \in \Sigma$. Then $\{Q_{\hat{\theta}}\}_{\hat{\theta} > 0}$ is a disintegration of P over $P_{\hat{\Theta}}$ consistent with $\hat{\Theta}$, condition $(Q_{\hat{\theta}})_{W_n} = \mathbf{Exp}(\hat{\theta})$ holds true for any $n \in \mathbb{N}$ and $\hat{\theta} > 0$, and the process W is $Q_{\hat{\theta}}$ -independent (see [10] Lemma 2.4). Thus due to [8], Proposition 4.4, we obtain that N is a P -MPP($\hat{\Theta}$). Clearly, all assumptions of Theorems 2.6 and 2.7 are satisfied and so are their conclusions. In particular, each of its assertions (i) to (iv) is valid.

In our next example the real-valued random variable $\hat{\Theta}$ is distributed according to the Lognormal law, a common choice in Reliability Theory.

Example 3.3 Let $G := \mathbb{R}$, let $\rho = \mathbf{N}(\mu, \sigma^2)$, with $(\mu, \sigma^2) \in \mathbb{R} \times (0, \infty)$ be a probability measure on \mathfrak{B} i.e.

$$\rho(B) := \int_B \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(t-\mu)^2}{2\sigma^2}} \cdot \chi_{\mathbb{R}}(t) \lambda(dt) \quad \text{for any } B \in \mathfrak{B}$$

and let $h : \mathbb{R} \mapsto \mathbb{R}$ be defined by $h(\theta) := e^\theta$ for any $\theta \in \mathbb{R}$. Fix on arbitrary $\theta \in \mathbb{R}$ and define the probability measures $Q_n(\theta)$ by means of $Q_n(\theta) := \mathbf{Exp}(h(\theta))$ for all $n \in \mathbb{N}$. Let (Ω, Σ, P) , Θ , N , W and $\{P_\theta\}_{\theta \in \mathbb{R}}$ be as in Example 3.1 with $G = \mathbb{R}$ and ρ in the place of μ .

Define the map $C \in \mathcal{L}^1(P_{h(\Theta)})$ by $C(h(\theta)) := h(\theta)$ for any $\theta \in \mathbb{R}$, and for any fixed $\theta \in \mathbb{R}$ define the density $f_{h(\theta)} := F'_{h(\theta)}$ by $f_{h(\theta)}(t) := h(\theta) \cdot e^{-h(\theta)t}$ for any $t > 0$. Clearly, for any fixed $\theta \in \mathbb{R}$, the density $f_{h(\theta)}$ is dominated by $C(h(\theta))$, and the function $p_h : \mathbb{R} \mapsto \mathcal{Y}$ defined by means of $p_h(\theta) := \lim_{t \rightarrow 0} f_{h(\theta)}(t) = h(\theta)$ for any $\theta \in \mathbb{R}$, is positive and injective; hence $\{P_\theta\}_{\theta \in \mathbb{R}}$, N and h satisfy Assumption 2.4.

Let $\hat{\Theta} := h \circ \Theta$ and put $Q_{\hat{\theta}}(E) := (P_{\bullet}(E) \circ h^{-1})(\hat{\theta})$ for any $\hat{\theta} > 0$ and $E \in \Sigma$. It then follows as in Example 3.2 that there exists a MPP($\hat{\Theta}$) with a lognormally distributed real-valued random variable $\hat{\Theta}$ and that all assumptions of Theorem 2.7 are satisfied and so are its conclusions. In particular, each of its assertions (i) to (iv) is valid.

4 Counter-examples

The next counter-examples show that there exist probability spaces and counting processes on them satisfying assertions (i), (ii) and (iv) but not assertion (iii) of Theorems 2.6 and 2.7.

Moreover, the assumptions of Theorem 2.6, concerning the perfectness of the measure P and the countability of Σ , are not valid, showing in this way that they are essential for the equivalence (i) \iff (iii). The same examples show that the assumption of Theorem 2.7 concerning the existence of a disintegration consistent with Θ is not valid; hence it is essential for the equivalence of (iii) with any of the assertions (i), (ii), (iv).

To present our counter-examples we need the following result

Lemma 4.1 *Let B be a subset of Ω with $P^*(B) = 1$ and $P_*(B) = 0$. Put $\Sigma_b := \sigma(\Sigma \cup \{B\})$ and define $R := P_b : \Sigma_b \longrightarrow [0, 1]$ by means of $R(\tilde{D}) := P^*(\tilde{D} \cap B)$ for any $\tilde{D} \in \Sigma_b$. Then there does not exist any d -dimensional random vector Ψ on Ω such that there exists a disintegration $\{R_{\psi}\}_{\psi \in \mathbb{R}^d}$ of R over R_{Ψ} consistent with Ψ . In particular, if Σ is countably generated then R is non-perfect.*

Proof. Assume, if possible, that there exists a d -dimensional random vector Ψ on Ω such that there exists a disintegration $\{R_{\psi}\}_{\psi \in \mathbb{R}^d}$ of R over R_{Ψ} consistent with Ψ . For any $\omega \in \Omega$ put

$$Q_{\omega}(E) := R_{\Psi(\omega)}(E)$$

for each $E \in \tilde{\Sigma}$.

Claim 1. *The family $\{Q_{\omega}\}_{\omega \in \Omega}$ is a subfield regular conditional probability for R over $R \upharpoonright \mathcal{F}$ with $\mathcal{F} = \sigma(\Psi)$.*

Proof. For the definition of a subfield regular conditional probability see [3], Section 2. Clearly for any fixed $\omega \in \Omega$ the set-function Q_{ω} is a probability measure on Σ_b , and for any fixed $F \in \mathcal{F}$ the function $\omega \longmapsto Q_{\omega}(F)$ is \mathcal{F} -measurable. Furthermore, for each $F \in \mathcal{F}$ and $E \in \Sigma_b$ we have

$$\begin{aligned} \int_F Q_{\omega}(E) R(d\omega) &= \int_F R_{\Psi(\omega)}(E) R(d\omega) = \int_F \mathbb{E}_R[\chi_E \mid \mathcal{F}](\omega) R(d\omega) \\ &= \int_F \chi_E dR = \int \chi_{F \cap E} dR; \end{aligned}$$

hence

$$(5) \quad \int_F Q_\omega(E) R(d\omega) = R(E \cap F)$$

where the second equality follows from the assumption that the restriction of $\{R_\psi\}_{\psi \in \mathbb{R}^d}$ is a disintegration of R over R_Ψ consistent with Ψ , together with [8], Lemma 2.5 (i). As a consequence, we get that $\{Q_\omega\}_{\omega \in \Omega}$ is a subfield regular conditional probability for \tilde{P} over $\tilde{P} \upharpoonright \mathcal{F}$. This completes the proof of Claim 1. \square

Claim 2. There exists a P -null set $N \in \mathcal{F}$ such that for each $A \in \mathcal{F}$ condition $Q_\omega(A) = 1$ holds true for any $\omega \in N^c \cap A$.

Proof. Since $\{R_\psi\}_{\psi \in \mathbb{R}^d}$ is a disintegration of R over R_Ψ consistent with Ψ we get

$$(6) \quad \forall A \in \mathcal{F} \quad \exists N_A \in \mathcal{F}_0 \quad \forall \omega \in A \cap N_A^c \quad [Q_\omega(A) = 1],$$

where \mathcal{F}_0 is the set of all P -null sets in \mathcal{F} . Notice that \mathcal{F} is countably generated since \mathfrak{B}_d is so. Let \mathcal{G} be a countable generator of \mathcal{F} . Without loss of generality we may and do assume that \mathcal{G} is closed under finite intersections. Since \mathcal{G} is a countable generator of \mathcal{F} condition (6) can be rewritten as

$$\forall n \in \mathbb{N} \quad \forall A_n \in \mathcal{G} \quad \exists N_{A_n} \in \mathcal{F}_0 \quad \forall \omega \in A_n \cap N_{A_n}^c \quad [Q_\omega(A_n) = 1].$$

So, setting $N := \bigcup_{n \in \mathbb{N}} A_n$ we get that $N \in \mathcal{F}$ with $P(N) = 0$. Let us denote now by \mathcal{D} the class of all sets $A \in \mathcal{F}$ such that $Q_\omega(A) = 1$ for each $\omega \in N^c \cap A$. Then it can be easily proven that \mathcal{D} is a Dynkin class; hence by the Monotone Class Theorem the claim follows. \square

But by condition (5) and our assumption that $P^*(B) = 1$, we get for every $F \in \mathcal{F}$ that

$$\int_F Q_\omega(B) R(d\omega) = R(F \cap B) = R(F) = \int_F \chi_F(\omega) R(d\omega),$$

implying that $R(D) = 0$, where $D := \{\omega \in \Omega : Q_\omega(B) \neq 1\}$

Put $E := D \cup N$. For any $\omega \in E^c$ we get $Q_\omega(\{\omega\}) = 1$ and $Q_\omega(B) = 1$; hence $Q_\omega(B \cap \{\omega\}) = 1$, implying $B \cap \{\omega\} \neq \emptyset$ or $\omega \in B$. Thus we get $E^c \subseteq B$ or equivalently $B^c \subseteq E$, implying $1 = P^*(B^c) \leq R(E)$; hence $R(E) = 1$, a contradiction. In particular, if Σ is countably generated then Σ_b is so; hence applying [3], Theorem 6, we deduce that R is non-perfect. \square

Remark 4.2 Let Ω be an uncountable Polish space and P a non-atomic Borel measure on $\Sigma := \mathfrak{B}(\Omega)$. It should be known that there always exists a set $B \subseteq \Omega$ such that $P^*(B) = 1$ and $P_*(B) = 0$. But since we could not find it in the literature, we insert a short proof for completeness sake: Let $(\Omega, \hat{\Sigma}, \hat{P})$ be the completion of (Ω, Σ, P) . Then $(\Omega, \hat{\Sigma}, \hat{P})$ is isomorphic to the Lebesgue probability space $([0, 1], \mathcal{L}([0, 1]), \lambda)$ (cf.

e.g. [4], Corollary 344K). By [2], Proposition 1.4.11, there exists a subset \tilde{A} of \mathbb{R} such that each Lebesgue measurable set that is included in \tilde{A} or \tilde{A}^c is a λ -null set. Put $A := [0, 1] \cap \tilde{A}$ and $A_1 := [0, 1] \cap \tilde{A}^c$.

The subsets A and A_1 cannot be both Lebesgue measurable, since if they were so, then we would get that $\lambda(A) = \lambda(A_1) = 0$, implying $0 = \lambda(A \cup A_1) = \lambda([0, 1]) = 1$, a contradiction. Thus, if A is non-Lebesgue measurable, we infer that $\lambda_*(A) = 0$ and $\lambda^*(A) = 1$. Without loss of generality we may and do assume that A is non-Lebesgue measurable. So, letting $f : [0, 1] \rightarrow \Omega$ be an isomorphism between the Lebesgue probability space on $[0, 1]$ and $(\Omega, \hat{\Sigma}, \hat{P})$, we get that $B := f(A)$ is the desired set.

Remark 4.3 Let P, Θ and $\{P_\theta\}_{\theta \in \mathcal{Y}}$ be as in Example 3.2. Fix on arbitrary $\theta \in \mathcal{Y}$ and put $\Sigma_0 := \{L \in \Sigma : P(L) = 0\}$ and $\Sigma_{0,\theta} := \{L \in \Sigma : P_\theta(L) = 0\}$.

(a) Since for any fixed $E \in \tilde{\Sigma}$ the function $\theta \mapsto \tilde{P}_\theta(E) := \otimes_{n \in \mathbb{N}} \mathbf{Exp}(h(\theta))(E)$ is continuous, it can be easily seen that $\Sigma_0 = \Sigma_{0,\theta}$ for every $\theta \in \mathcal{Y}$, implying that $P^*(B) = P_\theta^*(B) = 1$ and $P_*(B) = (P_\theta)_*(B) = 0$. Thus, the probability measure P_θ can be extended to the probability measure $P_{\theta,b} : \Sigma_b \rightarrow [0, 1]$, defined by means of $P_{\theta,b}(\tilde{D}) := P_\theta^*(\tilde{D} \cap B)$ for any $\tilde{D} \in \Sigma_b$. Then for any fixed $D \in \Sigma_b$ the function $\theta \mapsto P_{\theta,b}(\tilde{D})$ is $\mathfrak{B}(\mathcal{Y})$ -measurable.

(b) For any $\theta \in \mathcal{Y}$ denote by $\hat{\Sigma}_\theta$ the completion of Σ with respect to P_θ . It then follows that $\hat{\Sigma} = \hat{\Sigma}_\theta$ for any $\theta \in \mathcal{Y}$; hence each completed probability measure \hat{P}_θ is defined on $\hat{\Sigma}$ and for any $E \in \hat{\Sigma}$ the function $\theta \mapsto \hat{P}_\theta(E)$ is $\mathfrak{B}(\mathcal{Y})$ -measurable.

Example 4.4 Let $(\Omega, \Sigma, P), N, \Theta, h$ and $\hat{\Theta}$ be as in Example 3.2. Then all the assumptions of Theorems 2.6 and 2.7 are satisfied, and so the equivalence of all assertions (under P) follows for each of the above two theorems. In particular, recall that each assertion of both theorems is valid.

Since by construction (Ω, Σ, P) is an uncountable non-atomic Polish probability space, it follows by Remark 4.2 that there exists a set $B \subseteq \Omega$ such that $P^*(B) = 1$ and $P_*(B) = 0$; hence we may define R and Σ_b as in Lemma 4.1. It can be easily seen that assertions (i) and (iv) of Theorems 2.6 and 2.7 remain valid under R , and taking into account Remark 4.3 so does assertion (ii). In particular, assertions (i), (ii) and (iv) of Theorem 2.7 are equivalent under R .

According to Lemma 4.1, there does not exist any real-valued random variable Ψ on Ω such that there exists a disintegration $\{R_\psi\}_{\psi \in \mathbb{R}}$ of R over R_Ψ consistent with Ψ , implying that assertion (iii) of Theorems 2.6 and 2.7 fails.

Note that, due to Lemma 4.1, neither the part of the Assumption 2.5 concerning the existence of a real-valued random variable Θ on Ω and a disintegration of $R := P_b$ over R_Θ consistent with Θ nor the perfectness assumption of Theorem 2.6 for the probability space (Ω, Σ_b, R) hold true. Thus, both assumptions are not necessary for the equivalences (i) \iff (ii) \iff (iv) and the perfectness assumption is essential for the equivalence (i) \iff (iii).

Concerning Theorem 2.7, due again to Lemma 4.1, the assumption of the existence of a disintegration of R over R_Θ consistent with Θ is not necessary for the equivalences $(i) \iff (ii) \iff (iv)$ but it is essential for the equivalence $(i) \iff (iii)$.

Example 4.5 Let (Ω, Σ, P) , N , Θ , h and $\hat{\Theta}$ be as in Example 3.2. Then all the assumptions of Theorems 2.6 and 2.7 are satisfied, and so the equivalence of all assertions (under P) follows for each of the above two theorems. In particular, recall that each assertion of both theorems is valid. Let $(\Omega, \hat{\Sigma}, \hat{P})$ be the completion of (Ω, Σ, P) . It can be easily seen that assertions (i) and (iv) of Theorems 2.6 and 2.7 remain valid under \hat{P} , and taking into account Remark 4.3 so does assertion (ii) . In particular, assertions (i) , (ii) and (iv) of Theorem 2.7 are equivalent under \hat{P} .

First notice that the probability measure \hat{P} is perfect since P is (cf. e.g. [5], Proposition 451G(c)(i)), but $\hat{\Sigma}$ is not countably generated; hence the countability assumption in Theorem 2.6 fails.

Claim. *There does not exist any real-valued random variable Ψ on Ω such that there exists a disintegration $\{Z_\psi\}_{\psi \in \mathbb{R}}$ of \hat{P} over \hat{P}_Ψ consistent with Ψ .*

Proof. Assume, if possible, that there exists Ψ on Ω such that there exists a disintegration $\{Z_\psi\}_{\psi \in \mathbb{R}}$ of \hat{P} over \hat{P}_Ψ consistent with Ψ . Fix on an arbitrary $A \in \hat{\Sigma}$ and define the function $S_\bullet(A) : \Omega \mapsto [0, 1]$ by means of

$$(S_\bullet(A))(\omega) := S_\omega(A) := (Z_\bullet(A) \circ \Psi)(\omega) \text{ for every } \omega \in \Omega.$$

Then by the same arguments as in the proof of Lemma 4.1 we get that $\{S_\omega\}_{\omega \in \Omega}$ is a subfield r.c.p. of \hat{P} over $\hat{P} \upharpoonright \sigma(\Psi)$. Since $\sigma(\Psi)$ is countably generated, it follows as in Lemma 4.1 that there exists a set $N \in \sigma(\Psi)$ such that $P(N) = 0$ and for any $A \in \sigma(\Psi)$ condition $S_\omega(A) = 1$ holds true for any $\omega \in N^c \cap A$. Choose a set $D \subseteq N^c$ such that $D \notin \sigma(\Psi)$ but $D \in \hat{\Sigma}$. Such a choice is possible, since the cardinality of $\sigma(\Psi)$ is \mathfrak{c} , \mathfrak{c} the cardinality of the continuum, while the cardinality of $\hat{\Sigma}$ is $2^{\mathfrak{c}}$. Then for each $\omega \notin N$ we obtain

$$1 = S_\omega(\{\omega\}) \leq S_\omega(D) \leq 1 \quad \text{if } \omega \in D$$

and

$$1 = S_\omega(\{\omega\}) \leq S_\omega(D^c) \leq 1 \quad \text{if } \omega \in D^c.$$

Thus, $D = N^c \cap \{\omega \in \Omega : S_\omega(D) = 1\} \in \sigma(\Psi)$, a contradiction. \square

As a consequence, it follows that neither Assumption 2.5 of Theorem 2.6 nor the assumption of Theorem 2.7 concerning the existence of a disintegration hold true. Moreover, the above claim yields that assertion (iii) of Theorems 2.6 and 2.7 fails; hence the countability assumption for Σ is essential for the equivalence of (i) and (iii) .

Remark 4.6 The above two counter-examples answer to the negative [10], Question 2.14, concerning the necessity of the assumptions of the existence of a disintegration of P over P_Θ consistent with Θ of Proposition 2.7 and Theorem 2.11 from [10], for the validity of their conclusions.

In fact, let $(\Omega, \hat{\Sigma}, \hat{P})$ and $N, h, \Theta, \hat{\Theta}$ and $\{P_\theta\}_{\theta>0}$ be as in Example 4.5, and let $\{\hat{P}_\theta\}_{\theta>0}$ be as in Remark 4.3. Then N is \hat{P} -eMRP($\mathbf{K}(h(\Theta))$) satisfying together with h and $\{\hat{P}_\theta\}_{\theta>0}$ Assumption 2.4, but according to the claim of Example 4.5, $\{\hat{P}_\theta\}_{\theta>0}$ can not be a disintegration of \hat{P} over \hat{P}_Θ consistent with Θ . For the counting process N the following are equivalent

- (a) N is a \hat{P} -MPP($\hat{\Theta}$);
- (b) N has the \hat{P} -multinomial property;
- (c) N has the \hat{P} -Markov property.

For the definition of the multinomial property we refer to e.g. [12], Section 2 page 2. Ad (a) \iff (b): N is a \hat{P} -MPP($\hat{\Theta}$) if and only if it is a \hat{P} -MPP($P_{\hat{\Theta}}$) (see Example 4.5) if and only if it has the \hat{P} -multinomial property (see [12], Theorem 4.2).

Ad (a) \iff (c): N is a \hat{P} -MPP($\hat{\Theta}$) if and only if it is a \hat{P} -MPP($\{\hat{P}_\theta\}_{\theta>0}, \nu$) (see Example 4.5). But the latter, due to [7], Theorem 3, is equivalent with the \hat{P} -Markov property for N (see also [10], Remark 2.8 (d) and Example 3.3).

Thus, in the above set-up we constructed a probability space $(\Omega, \hat{\Sigma}, \hat{P})$, a family of probability measure $\{\hat{P}_\theta\}_{\theta>0}$ on $\hat{\Sigma}$, and a counting process N being a \hat{P} -eMRP($\mathbf{K}(h(\Theta))$) satisfying together with h and $\{\hat{P}_\theta\}_{\theta>0}$ Assumption 2.4 such that the conclusions of Theorem 2.11 from [10], hold true but $\{\hat{P}_\theta\}_{\theta>0}$ is not necessarily a disintegration of \hat{P} over \hat{P}_Θ consistent with Θ .

References

- [1] ALBRECHT, P. : *Über einige Eigenschaften des gemischten Poissonprozesses*, Mitt. Ver. Schweiz. Vers. Math. 81, 241-250. (1981).
- [2] COHN, D.L. : *Measure Theory 2nd edition*, Birkhäuser Advanced Texts. (2013).
- [3] FADEN, A.M. : The existence of regular conditional probabilities: Necessary and sufficient conditions, *Ann. Probab.* **13** (No. 1), 288-298.(1985).
- [4] FREMLIN, D.H. : *Measure Theory, Vol. 3*. Torres Fremlin (Ed.). (2003).
- [5] FREMLIN, D.H. : *Measure Theory, Vol. 4*. Torres Fremlin (Ed.). (2003).
- [6] GRANDELL, J. : *Mixed Poisson Processes*, Chapman & Hall. (1997).
- [7] HUANG, W.J. : *On the Characterization of Point Processes with the Exchangeable and Markov Properties*, Sankhya, Volume 52, Series A, Pt. 1, pp. 16-27. (1990).

- [8] LYBEROPOULOS, D.P. AND MACHERAS, N.D. : *Some characterizations of mixed Poisson processes*, , Sankhya, Volume 74, Series A, Pt. 1, pp. 57-79. (2012).
- [9] LYBEROPOULOS, D.P. AND MACHERAS, N.D. : *Some characterizations and a construction of mixed renewal processes*, arXiv:1205.4441. (2014).
- [10] MACHERAS, N.D. AND TZANINIS, S.M. : *Some characterizations for Markov processes as mixed renewal processes*, arXiv:1407.3072. (2016).
- [11] SCHMIDT, K.D. : *Lectures on Risk Theory*, B.G. Teubner, Stuttgart. (1996).
- [12] SCHMIDT, K.D. AND ZOCHER, M. : *Claim Number Processes having the Multinomial Property*. (2011).
<http://www.math.tu-dresden.de/sto/schmidt/dsvm/dsvm2003-1.pdf>
- [13] SERFOZO, R.F. : *Conditional Poisson Processes*, Journal of Applied Probability, Vol. 9, No. 2 , Issue 1, pp 288-302. (1972).
- [14] SERFOZO, R.F. : *Processes with Conditional Stationary Independent Increments*, Journal of Applied Probability, Vol. 9, No. 2 , Issue 1, pp 303-315. (1972).

D.P. LYBEROPOULOS, N.D. MACHERAS AND S.M TZANINIS
 DEPARTMENT OF STATISTICS AND INSURANCE SCIENCE
 UNIVERSITY OF PIRAEUS, 80 KARAOLI AND DIMITRIOU STREET
 185 34 PIRAEUS, GREECE

E-mail: dilyber@webmail.unipi.gr, macheras@unipi.gr AND stzaninis@unipi.gr